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An $O(3, 1)$ nonlinear σ-model and the Ablowitz–Ladik hierarchy

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Abstract. A solvable nonlinear 2D field model is proposed which is shown to be closely connected with the nonlinear Schrödinger chain. The conservation laws and soliton solutions for the corresponding field equations are obtained.

1. Introduction

The distinguishing feature of nonlinear σ-models is the fact that the nonlinearity in these models is geometric in character: it is caused not by unharmonic potentials or interactions but by constraints imposed on the system. Formally, the constraints manifest themselves in the fact that the fields are taken to satisfy some additional conditions and, hence, become valued in some nonlinear manifold.

A good example of a situation in which the restrictions imposed on a linear system lead to a non-trivial highly nonlinear distribution of the field is the model proposed by Pohlmeyer [1], who considered the wave equation restricted to the $O(N)$-invariant manifold. Later, the $O(4)$ model was studied by Lund [2], who developed the corresponding variant of the inverse-scattering method (ISM) (see also the work by Getmanov [3]). An example of the appearance of such a model in physics was demonstrated by Lund and Regge [4].

The present work is devoted to the case somewhat opposite to the one studied in [1]. While Pohlmeyer deals with the Lorentz-invariant differential operator and the Dalambertian and 'Euclidean' restrictions, we will consider the Euclidean analogue of the wave operator, the Laplacian and the Lorentz-invariant restrictions. Thus, the model presented below can be termed an $O(3, 1)$-invariant nonlinear σ-model.

So, consider the problem

$$\Delta\phi^{\mu} + \lambda\phi^{\mu} = 0 \tag{1.1}$$

under the restriction

$$\phi_{\mu}\phi^{\mu} = -1. \tag{1.2}$$

Here $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and ϕ^{μ} is a space-like vector from the Minkowski space

$$\phi^{\mu} = (\phi^0, \phi^1, \phi^2, \phi^3) \tag{1.3}$$

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with the scalar product

$$\phi_\mu \psi^\mu = \phi^0\psi^0 - \phi^1\psi^1 - \phi^2\psi^2 - \phi^3\psi^3. \tag{1.4}$$

To satisfy condition (1.2), the Lagrange multiplier λ is set to

$$\lambda = -(\nabla\phi_\mu, \nabla\phi^\mu). \tag{1.5}$$

Hereafter, the scale invariance will be broken by normalizing coordinates in such a way that

$$\partial_x\phi_\mu\, \partial_x\phi^\mu - \partial_y\phi_\mu\, \partial_y\phi^\mu = m^2 \qquad \partial_x\phi_\mu\, \partial_y\phi^\mu = 0 \tag{1.6}$$

where m is a constant playing the role of the spatial scale (∂_x stands for ∂/∂_x, etc).

The vectors ϕ^μ, $\partial_x\phi^\mu$, $\partial_y\phi^\mu$ together with the time-like vector χ^μ which is normal to the surface $\phi^\mu(x, y)$

$$\chi_\mu\phi^\mu = \chi_\mu\, \partial_x\phi^\mu = \chi_\mu\, \partial_y\phi^\mu = 0 \qquad \chi_\mu\chi^\mu = 1 \tag{1.7}$$

form a local basis in Minkowski space.

To demonstrate the relationship between the model considered and the $O(4)$ σ-model, as well as with the sine–Gordon equation, consider the quantity α, defined by

$$m^2\cos 2\alpha = (\nabla\phi_\mu, \nabla\phi^\mu). \tag{1.8}$$

After some straightforward calculations which we have omitted here, one can obtain the Gauss–Weingarten system

$$\partial_x \begin{pmatrix} \phi^\mu \\ \partial_x\phi^\mu \\ \partial_y\phi^\mu \\ \chi^\mu \end{pmatrix} = U \begin{pmatrix} \phi^\mu \\ \partial_x\phi^\mu \\ \partial_y\phi^\mu \\ \chi^\mu \end{pmatrix} \qquad \partial_y \begin{pmatrix} \phi^\mu \\ \partial_x\phi^\mu \\ \partial_y\phi^\mu \\ \chi^\mu \end{pmatrix} = V \begin{pmatrix} \phi^\mu \\ \partial_x\phi^\mu \\ \partial_y\phi^\mu \\ \chi^\mu \end{pmatrix} \tag{1.9}$$

where the 4×4 matrices U and V are given by

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ m^2\cos^2\alpha & -\alpha_x\tan\alpha & -\alpha_y\cot\alpha & mu \\ 0 & -\alpha_y\tan\alpha & \alpha_x\cot\alpha & mv \\ 0 & -u/(m\cos^2\alpha) & v/(m\sin^2\alpha) & 0 \end{pmatrix} \tag{1.10}$$

$$V = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -\alpha_y\tan\alpha & \alpha_x\cot\alpha & mv \\ -m^2\sin^2\alpha & \alpha_x\tan\alpha & \alpha_y\cot\alpha & -mu \\ 0 & -v/(m\cos^2\alpha) & -u/(m\sin^2\alpha) & 0 \end{pmatrix} \tag{1.11}$$

with u and v being some functions of x and y.

The integrability conditions for the system (1.9), the so-called Gauss–Codazzi equations, can be written in terms of α, u and v (see 1.2) as follows

$$\Delta\alpha - m^2\sin\alpha\cos\alpha + \frac{u^2+v^2}{\sin\alpha\cos\alpha} = 0 \tag{1.12}$$

$$(u\cot\alpha)_x + (v\cot\alpha)_y = 0 \tag{1.13}$$

$$(u\tan\alpha)_y - (v\tan\alpha)_x = 0. \tag{1.14}$$

After integrating equation (1.14)

$$u = \beta_x \cot\alpha \qquad v = \beta_y \cot\alpha \tag{1.15}$$

the remaining two equations, (1.12) and (1.13), become

$$\Delta\alpha - m^2 \sin\alpha\cos\alpha + \frac{\cos\alpha}{\sin^3\alpha}(\nabla\beta, \nabla\beta) = 0 \tag{1.16}$$

and

$$\operatorname{div}(\cot^2\alpha\nabla\beta) = 0. \tag{1.17}$$

Note that the particular case ($\beta = 0$) of the above equations is the elliptic sine–Gordon equation which was studied by Leibbrandt [5] and which naturally appears when the problem of surface embedding in the Minkowski space is considered [6].

These equations are the Euler–Lagrange equations for the action

$$S = \iint \mathrm{d}x\,\mathrm{d}y\,\mathcal{L} \tag{1.18}$$

with the Lagrangian

$$\mathcal{L} = (\nabla\alpha, \nabla\alpha) + \cot^2\alpha(\nabla\beta, \nabla\beta) - m^2\cos^2\alpha \tag{1.19}$$

which can be rewritten in terms of the function

$$q = \cos\alpha\exp(\mathrm{i}\beta) \tag{1.20}$$

as follows

$$\mathcal{L} = \frac{(\nabla q, \nabla\bar{q})}{1 - |q|^2} - m^2|q|^2 \tag{1.21}$$

or, after rescaling $x \to \frac{2}{m}x$, $y \to \frac{2}{m}y$, as

$$\mathcal{L} = \frac{(\nabla q, \nabla\bar{q})}{1 - |q|^2} - 4|q|^2. \tag{1.22}$$

Here, I want to note that the Lagrangian of the problem (1.1), (1.2), $\mathcal{L} = (\nabla\phi_\mu, \nabla\phi^\mu)$, which consists of the 'kinetic energy' only, is transformed in the framework of Pohlmeyer's method to one with a 'mass' term (the last term in (1.19) or (1.21)). Such an effect has been known since the studies of the $O(3)$ σ-model $\mathcal{L} = (\nabla\phi, \nabla\phi)$, $|\phi| = 1$), which can be reduced (see, e.g., [1]) to the sine–Gordon system with the Lagrangian possessing the 'potential-energy' term (it is similar to (1.19) with the β-term omitted). An interesting problem arises from the fact that the traditional way of taking into account the constraint $|\phi| = 1$, using the stereographic projection, leads again to the pure 'kinetic' Lagrangian $\mathcal{L} = (\nabla q, \nabla\bar{q})/(1 + |q|^2)^2$. The problem is to describe the $O(3,1)$ model in an analogous way using the group parameters, but this question will not be discussed in the present paper.

The field model described by the Lagrangian (1.22) can also be obtained as the reduction of the Euclidean version of the principal chiral-field model, as outlined in the appendix.

The field equations corresponding to the Lagrangian (1.22) are the main subject of the present work. It is possible to develop the inverse-scattering scheme applicable to this equation (as achieved for the $O(4)$ model by Lund [2]). However, in the present work, use will be made of another approach which is based on the remarkable fact that the model considered, as will be shown below, is closely related to the hierarchy of integrable equations proposed by Ablowitz and Ladik [7] (some features of this hierarchy are discussed briefly in section 3). It turns out that this relationship provides almost all the results that are usually obtained in the framework of the ISM. It can be used to demonstrate that the $O(3,1)$ σ-model possesses an infinite number of symmetries and conserved quantities (see section 4) and to obtain soliton and some other solutions for the field equation (see section 5). Although not all problems arising in connection with the $O(3,1)$ σ-model can be solved using this approach, it seems to be rather interesting and may be useful in some situations (some remarks and examples on this question can be found in section 6). As to the ISM, the elaboration of the corresponding inverse-scattering scheme is surely a problem of primary importance, but this question, which is worth special consideration, will not be featured here but will be discussed by the author in a future paper.

2. Field equation

The field equation corresponding to the Lagrangian (1.22) can be written as

$$\Delta q + \frac{\bar{q}}{p}(\nabla q, \nabla q) + 4pq = 0 \tag{2.1}$$

where

$$p \equiv 1 - |q|^2 \tag{2.2}$$

or, by introducing the complex variables

$$z = x + \mathrm{i}y \qquad \bar{z} = x - \mathrm{i}y \tag{2.3}$$

as follows

$$q_{z\bar{z}} + \frac{\bar{q}}{p} q_z q_{\bar{z}} + pq = 0. \tag{2.4}$$

Some simple transformations lead to the following representation of equation (2.4):

$$\hat{D}_+ \hat{D}_- q = \hat{D}_- \hat{D}_+ q = q \tag{2.5}$$

where the nonlinear operators $\hat{D}_\pm$ are defined by

$$\hat{D}_+ q \equiv \frac{\mathrm{i}}{1 - |q|^2} \partial_z q \qquad \hat{D}_- q \equiv \frac{\mathrm{i}}{1 - |q|^2} \partial_{\bar{z}} q \tag{2.6}$$

(the commutator $[\hat{D}_+, \hat{D}_-]$, which is, in general, non-zero, vanishes when applied to a solution of (2.4)).

Noting that if q is a solution of (2.4) then $\hat{D}_+ q$ and $\hat{D}_- q$ are also solutions, one can construct, starting from q, a double-infinite (in a general case) sequence $\{q_n\}$ by means of the recurrence

$$q_{n\pm 1} = \hat{D}_\pm q_n \tag{2.7}$$

and deal with, instead of (2.4), the two infinite systems

$$\mathrm{i}\partial_z q_n = p_n q_{n+1} \tag{2.8}$$

$$\mathrm{i}\partial_{\bar{z}} q_n = p_n q_{n-1}. \tag{2.9}$$

Since systems (2.8) and (2.9) can be treated separately, one can say that such an approach reduces a partial differential equation (2.4) to the two systems of ordinal differential equations, which in the next section will be shown to belong to the Ablowitz–Ladik hierarchy.

3. The Ablowitz–Ladik hierarchy

The Ablowitz–Ladik hierarchy—the discrete version of the AKNS hierarchy—is the infinite set of ordinal differential equations, the most well known of these being the discrete nonlinear Schrödinger equation (DNLSE)

$$\mathrm{i}\dot{q}_n = q_{n-1} - 2q_n + q_{n+1} + \kappa|q_n|^2(q_{n-1} + q_{n+1}) \tag{3.1}$$

with the dot standing for the derivative with respect to time and $\kappa = \pm 1$, which, after the substitution $q_n \to q_n \exp(2\mathrm{i}t)$, takes the form

$$\mathrm{i}\dot{q}_n = (1 + \kappa|q_n|^2)(q_{n-1} + q_{n+1}). \tag{3.2}$$

The inverse scattering method for the infinite chain ($-\infty < n < \infty$) was developed in [8] for $\kappa = 1$ and in [9] for $\kappa = -1$. The problem (3.1) under quasiperiodic conditions was solved in [10, 11] (this case will not be discussed in what follows). Recently, the author investigated the corresponding finite system (3.2) ($\kappa = -1$; $n = 1, \ldots, N$; $|q_0| = |q_{N+1}| = 1$) [12]. In all the above-mentioned versions of the problem related to (3.1), the DNLSE chain turns out to be integrable: it possesses a sufficient number (infinite in the infinite case and N in the case of an N-node chain) of first integrals of motion I_m ($\dot{I}_m = 0$; see, e.g., [7]) in involution.

The system (3.2) (hereafter, only the case of $\kappa = -1$ will be considered) is Hamiltonian and can be rewritten as

$$\dot{q}_n = \{H, q_n\} \tag{3.3}$$

with Poisson brackets defined by

$$\{q_m, \bar{q}_n\} = \mathrm{i}p_n\delta_{mn} \tag{3.4}$$

$$\{q_m, q_n\} = \{\bar{q}_m, \bar{q}n\} = 0. \tag{3.5}$$

Here

$$p_n \equiv 1 - |q_n|^2 \tag{3.6}$$

and the Hamiltonian

$$H = 2\,\mathrm{Re}\, I_1 \tag{3.7}$$

where I_1 is the first integral given by

$$I_1 = \sum \bar{q}_n q_{n+1} \tag{3.8}$$

(the summation is performed over all n's in the infinite case and from 0 to N in the finite case).

The geometrical interpretation of the evolution equation (3.3) is that it describes the flow over the manifold determined by the set of integrals I_m. Each of the constants of motion can play the part of the Hamiltonian, giving rise to different flows over the same manifold

$$\frac{\partial q_n}{\partial t_k} = \{H_k, q_n\} \tag{3.9}$$

where $H_k = H_k(I_1, I_2, \ldots)$. The Ablowitz–Ladik hierarchy is the set of equations (3.9) which may be termed 'higher DNLSE's'.

The flows (3.9) commute since the corresponding integrals are in involution: $\{I_l, I_m\} = 0$. So we can treat them simultaneously, regarding (3.9) as a system of equations.

Let us consider two of the above flows: one determined by H and one with the Hamiltonian G given by

$$G = -2\,\mathrm{Im}\, I_1 \tag{3.10}$$

which leads to the system

$$\frac{\partial q_n}{\partial x} = \{H, q_n\} = -\mathrm{i} p_n (q_{n+1} + q_{n-1}) \tag{3.11}$$

$$\frac{\partial q_n}{\partial y} = \{G, q_n\} = p_n (q_{n+1} - q_{n-1}). \tag{3.12}$$

Since $\{H, G\} = 0$, this system is compatible and can be solved using a standard technique (some of its solutions will be written below). Differentiating (3.11) with respect to x and (3.12) with respect to y, one can obtain the identity

$$\mathrm{div}\, \frac{1}{p_n} \nabla q_n + 2(p_{n-1} + p_{n+1}) q_n = 0 \tag{3.13}$$

from which, again using (3.11) and (3.12), one can derive the following identity:

$$\mathrm{div}\, \frac{1}{p_n} \nabla q_n + \left[4 - \frac{1}{p_n^2} |\nabla q_n|^2\right] q_n = 0 \tag{3.14}$$

which is nothing more than equation (2.1).

In this way, we obtained the following result: each solution of system (3.11) and (3.22), closely related to the DNLSE (3.2), is a solution to the Euler–Lagrange equation for the $O(3,1)$ σ-model (1.18), (1.22). Also, it is not surprising, since the system (3.11), (3.12), when rewritten in terms of $z = x + \mathrm{i}y$ and $\bar{z}$, is exactly the same as system (2.8), (2.9).

In the following, use will be made of some of the higher DNLSE

$$\frac{\partial q_n}{\partial z_m} = \{I_m, q_n\} \tag{3.15}$$

and

$$\frac{\partial q_n}{\partial \bar{z}_m} = \{\bar{I}_m, q_n\} \tag{3.16}$$

for $m = 2, 3$ with the integrals $I_{2,3}$ being given by

$$I_2 = \sum \bar{q}_{n-1} p_n q_{n+1} - \tfrac{1}{2}\bar{q}_{n-1}^2 q_n^2 \tag{3.17}$$

$$I_3 = \sum \bar{q}_{n-2} p_{n-1} p_n q_{n+1} - \bar{q}_{n-1}\bar{q}_n p_n q_{n+1}^2 - \bar{q}_{n-1}^2 p_n q_n q_{n+1} + \tfrac{1}{3}\bar{q}_n^3 q_{n+1}^3. \tag{3.18}$$

4. Symmetries and conserved currents

Presenting the Euler–Lagrange equations (2.4) in the form

$$\begin{pmatrix} \Delta_1[q,\bar{q}] \\ \Delta_2[q,\bar{q}] \end{pmatrix} = 0 \tag{4.1}$$

where

$$\Delta_1[q,r] \equiv \frac{q_{z\bar{z}}}{1-qr} + \frac{q_z q_{\bar{z}} r}{(1-qr)^2} + q \tag{4.2}$$

and $\Delta_2[q,r] = \Delta_1[r,q]$, one can write the equation determining symmetries as

$$\hat{K}[q,\bar{q}]\begin{pmatrix} Q \\ R \end{pmatrix} = 0 \tag{4.3}$$

where $\hat{K}$ is the Frechet derivative of the Euler–Lagrange operator:

$$\hat{K}[q,r]\begin{pmatrix} Q \\ R \end{pmatrix} = \lim_{\epsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\begin{pmatrix} \Delta_1[q+\epsilon Q, r+\epsilon R] \\ \Delta_2[q+\epsilon Q, r+\epsilon R] \end{pmatrix}. \tag{4.4}$$

It is obvious that the derivatives

$$Q = \frac{\partial q}{\partial \lambda} \qquad R = \frac{\partial \bar{q}}{\partial \lambda} \tag{4.5}$$

with respect to any parameter λ solve equation (4.3). The Ablowitz–Ladik hierarchy provides an infinite set of differentiations (see (3.15) and (3.16)) which can be used to obtain an infinite number of symmetries:

$$\mathcal{Q}_j \equiv \begin{pmatrix} Q_j \\ R_j \end{pmatrix} = \frac{\partial}{\partial z_j}\begin{pmatrix} q \\ \bar{q} \end{pmatrix} \qquad j = 0, \pm 1, \pm 2, \ldots \tag{4.6}$$

where $z_{-j} \equiv \bar{z}_j$.

The simplest of these

$$Q_0 = \begin{pmatrix} iq \\ -i\bar{q} \end{pmatrix} \tag{4.7}$$

corresponds to the invariance of the problem under the transformations $q \to qe^{i\epsilon}$, $\bar{q} \to \bar{q}e^{-i\epsilon}$. The next symmetries,

$$Q_1 = \begin{pmatrix} q_z \\ \bar{q}_z \end{pmatrix} \qquad Q_{-1} = \begin{pmatrix} q_{\bar{z}} \\ \bar{q}_{\bar{z}} \end{pmatrix} \tag{4.8}$$

correspond to a shift of coordinates and stem from the autonomity of our system. The above symmetries $Q_{0,\pm 1}$ are 'usual' and inherent to a wide range of models of physical interest, while existence of the 'higher' symmetries $Q_{\pm j}$, $j \geqslant 2$, is a remarkable feature of the model considered and indicates its integrability. The first few of these can be written as

$$Q_2 = \begin{pmatrix} \mathrm{i}\left(q_{zz} + \dfrac{2}{p} q q_z \bar{q}_z\right) \\ -\mathrm{i}\left(\bar{q}_{zz} + \dfrac{2}{p} q_z \bar{q}_z \bar{q}\right) \end{pmatrix} \tag{4.9}$$

$$Q_3 = \begin{pmatrix} -q_{zzz} - \dfrac{3}{p}(q q_{zz} \bar{q}_z + q_z^2 \bar{q}_z) - \dfrac{3}{p^2} q^2 q_z \bar{q}_z^2 \\ -\bar{q}_{zzz} - \dfrac{3}{p}(q_z \bar{q}_{zz} \bar{q} + q_z \bar{q}_z^2) - \dfrac{3}{p^2} q_z^2 \bar{q}_z \bar{q}^2 \end{pmatrix} \tag{4.10}$$

(the expressions for their counterparts $Q_{-2,-3}$ can be obtained from (4.9) and (4.10) by complex conjugation: $Q_{-j} = \bar{R}_j$, $R_{-j} = \bar{Q}_j$).

Each of the symmetries (4.6) gives rise to the conservation law $Q_j \Delta_2 + R_j \Delta_1 = \partial_z J_j' + \partial_{\bar{z}} J_j''$ (see [13]). The first few that correspond to the symmetries $Q_0, \ldots, Q_3$ can be written as

$$\partial_z \frac{q_{\bar{z}} \bar{q}}{p} - \partial_{\bar{z}} \frac{q \bar{q}_z}{p} = 0 \tag{4.11}$$

$$\partial_z |q|^2 + \partial_{\bar{z}} \frac{q_z \bar{q}_z}{p} = 0 \tag{4.12}$$

$$\partial_z [q_z \bar{q} - q \bar{q}_z] + \partial_{\bar{z}} \left[\frac{1}{p}(q_{zz} \bar{q}_z - q_z \bar{q}_{zz}) + \frac{q_z \bar{q}_z}{p^2}(q \bar{q}_z - q_z \bar{q})\right] = 0 \tag{4.13}$$

$$\partial_z \left[(2p-1)\frac{q_z \bar{q}_z}{p}\right] + \partial_{\bar{z}} \left[\frac{q_{zz} \bar{q}_{zz}}{p} - \frac{p_{zz} q_z \bar{q}_z}{p^2} + (1-4p)\frac{q_z^2 \bar{q}_z^2}{p^3}\right] = 0. \tag{4.14}$$

Returning from z and $\bar{z}$ coordinates to x and y coordinates, they can be rewritten in the divergent form

$$\operatorname{div} J^{(m)} = 0. \tag{4.15}$$

So, the current corresponding to the phase invariance, $J^{(0)}$, is given by

$$J^{(0)} = \frac{\bar{q} \nabla q - q \nabla \bar{q}}{\mathrm{i}p} = \frac{2|q|^2}{1 - |q|^2} \nabla \arg q. \tag{4.16}$$

The currents corresponding to $\mathcal{Q}_{\pm 1}$

$$J^{(1)} = \begin{pmatrix} \frac{1}{4p}(q_x\bar{q}_x - q_y\bar{q}_y) - p \\ \frac{1}{4p}(q_x\bar{q}_y + q_y\bar{q}_x) \end{pmatrix} \qquad J^{(-1)} = -\begin{pmatrix} \frac{1}{4p}(q_x\bar{q}_y + q_y\bar{q}_x) \\ \frac{1}{4p}(-q_x\bar{q}_x + q_y\bar{q}_y) - p \end{pmatrix} \tag{4.17}$$

constitute an Euclidean version of the energy–momentum tensor

$$T = (T_{ij}) = \left(\frac{\partial\mathcal{L}}{\partial q_{x_i}}q_{x_j} + \frac{\partial\mathcal{L}}{\partial\bar{q}_{x_i}}\bar{q}_{x_j} - \delta_{ij}\mathcal{L}\right) = 4(J^{(1)}, -J^{(-1)}) + 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.18}$$

(here $x_1 = x$, $x_2 = y$).

It is tempting to present the conservation laws (4.15) in the 'curl' form

$$J^{(j)} = \mathrm{curl}_2\, C^{(j)} \tag{4.19}$$

where

$$\mathrm{curl}_2\, C \equiv \begin{pmatrix} \partial_y C \\ -\partial_x C \end{pmatrix}. \tag{4.20}$$

The 'potentials' $C^{(j)}$, which are non-local in the general case, can be constructed in the framework of the DNLSE chain (2.8), (2.9). This means that, instead of a current $J^{(j)} = J^{(j)}[q]$, one should consider, for a given j, the infinite set of currents

$$J_n^{(j)} = J^{(j)}[q_n] \tag{4.21}$$

where $\{q_n\}$ is the sequence of the solutions of the field equation (2.4) defined by (2.7), i.e. the solution of system (2.8), (2.9). In terms of this sequence, $J^{(0)}$ can be rewritten as

$$J_n^{(0)} = \begin{pmatrix} -q_{n-1}\bar{q}_n - q_n\bar{q}_{n+1} - \bar{q}_{n-1}q_n - \bar{q}_n q_{n+1} \\ \mathrm{i}(q_{n-1}\bar{q}_n + q_n\bar{q}_{n+1}) - \mathrm{i}(\bar{q}_{n-1}q_n + \bar{q}_n q_{n+1}) \end{pmatrix} = \mathrm{curl}_2\, C_n^{(0)}. \tag{4.22}$$

One can obtain from (2.8) and (2.9) the following expression for the potential $C_n^{(0)}$:

$$C_n^{(0)} = {\sum_{m=-\infty}^{n}}' \ln p_{m-1}p_m. \tag{4.23}$$

Analogously, for the currents $J^{(\pm 1)}$, which are given, in 'node' representation, by

$$\begin{aligned} J_n^{(1)} &= \frac{p_n}{2}\begin{pmatrix} \bar{q}_{n-1}q_{n+1} + q_{n-1}\bar{q}_{n+1} - 2 \\ \mathrm{i}(\bar{q}_{n-1}q_{n+1} - q_{n-1}\bar{q}_{n+1}) \end{pmatrix} \\ J_n^{(-1)} &= \frac{p_n}{2}\begin{pmatrix} \mathrm{i}(q_{n-1}\bar{q}_{n+1} - \bar{q}_{n-1}q_{n+1}) \\ q_{n-1}\bar{q}_{n+1} + \bar{q}_{n-1}q_{n+1} + 2 \end{pmatrix} \end{aligned} \tag{4.24}$$

the potentials can be written as

$$C_n^{(1)} = \mathrm{Re}\,{\sum_{m=-\infty}^{n}}' \bar{q}_{m-1}q_m \qquad C_n^{(-1)} = \mathrm{Im}\,{\sum_{m=-\infty}^{n}}' \bar{q}_{m-1}q_m. \tag{4.25}$$

The primes in (4.23) and (4.25) denote that the corresponding sums should be properly regularized if necessary.

5. Solutions for the field equation

The relation between the $O(3,1)$ σ-model and the Ablowitz–Ladik hierarchy provides us with a wide range of solutions for the field equation (2.4), which can be obtained by modification of the *already known* solutions of the DNLSE chain (3.2). In the present paper, I will not discuss stationary and quasiperiodic solutions and confine myself to solutions corresponding to the dark solitons [9] and to the 'finite nodes' [12] of the DNLSE.

The dark solitons of the DNLSE can be presented in the following form (derivation of the corresponding formulae can be found in [9]):

$$q_n = \rho \exp(-2\mathrm{i}r^2 x) \frac{\det \| h_k(n)\delta_{jk} + D_{jk} \mathrm{e}^{\mathrm{i}(\alpha_j+\alpha_k)} \|_{j,k=1,\dots,N}}{\det \| h_k(n)\delta_{jk} + D_{jk} \|_{j,k=1,\dots,N}} \tag{5.1}$$

where $h_k(n)$, ($h_k(n) = h_k(0)|\zeta_k|^{-2n}$) depend on x (x in (5.1) stands for t in (3.2)),

$$D_{jk} = (1 - \zeta_j \bar{\zeta}_k)^{-1} \tag{5.2}$$

and $\alpha_k = (\pi/2) + \arg(r - \zeta_k^{-1})$. The parameters ζ_k, which are the eigenvalues of the corresponding scattering problem (see [9]), are located on the arc $|r\zeta - 1| = \rho$. The constants ρ, which characterize q at infinity, and r are related by $\rho^2 + r^2 = 1$.

Although the expression (5.1) was obtained for equation (3.2), it can be shown that the corresponding solution for the system (2.8), (2.9) has the same structure with the elements h_k depending on $z, \bar{z}$ as follows

$$h_k \propto \exp\{2r(|\zeta_k|^{-2} - 1) \operatorname{Im} \zeta_k z\} \tag{5.3}$$

which leads to the following expression for the N-soliton solution for the $O(3,1)$ σ-model:

$$q_N^{(s)}(x,y) = \rho \exp(-2\mathrm{i}r^2 x) \frac{\det \| h_k(x,y)\delta_{jk} + D_{jk} \mathrm{e}^{\mathrm{i}(\alpha_j+\alpha_k)} \|_{j,k=1,\dots,N}}{\det \| h_k(x,y)\delta_{jk} + D_{jk} \|_{j,k=1,\dots,N}} \tag{5.4}$$

where

$$h_k(x,y) = h_k^0 \exp(-\lambda_k x - \mu_k y) \tag{5.5}$$

with

$$\begin{pmatrix} \lambda_k \\ \mu_k \end{pmatrix} = \frac{4\rho(\rho + \cos\psi_k)}{1 + 2\rho\cos\psi_k + \rho^2} \begin{pmatrix} \rho \sin\psi_k \\ 1 + \rho\cos\psi_k \end{pmatrix}. \tag{5.6}$$

Here h_k^0 and ψ_k, which are related to ζ_k through $r\zeta_k = 1 + \rho\exp(\mathrm{i}\psi_k)$, are arbitrary constants.

The expression for the modulus of $q_N^{(s)}$ can be written as

$$|q_N^{(s)}|^2 = 1 - r^2 \frac{A_N^+ A_N^-}{A_N^2} \tag{5.7}$$

where

$$A_N(x,y) \equiv \det \| h_k(x,y)\delta_{jk} + D_{jk} \|_{j,k=1,\dots,N} \tag{5.8}$$

and $A_N^{\pm}$ are given by an expression similar to (5.8) with h_k replaced by $h_k|\zeta_k|^{\pm 2}$. It can be seen that the solutions obtained satisfy the so-called 'finite-density' boundary conditions

$$|q_N^{(s)}| \to \rho \qquad \text{as } |x|, |y| \to \infty \tag{5.9}$$

and may be termed the dark-soliton solution for the $O(3, 1)$ σ-model.

The one-soliton solution can be written as

$$q_1^{(s)}(x, y) = \rho \exp(-2\mathrm{i}r^2 x) \frac{1 + \exp(\lambda x + \mu y + 2\mathrm{i}\vartheta)}{1 + \exp(\lambda x + \mu y)} \tag{5.10}$$

where

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = 4\rho \sin\vartheta \begin{pmatrix} -\rho\cos\vartheta \\ [1 - \rho^2\cos^2\vartheta]^{1/2} \end{pmatrix}. \tag{5.11}$$

The modulus of $q_1^{(s)}$ is given by

$$|q_1^{(s)}(x, y)|^2 = \rho^2 - \frac{\rho^2 \sin^2\vartheta}{\cosh^2\left(\frac{\lambda}{2}x + \frac{\mu}{2}y\right)}. \tag{5.12}$$

It is clearly seen that this soliton can be viewed as a hole against a ρ-valued background (this is why such solitons have been termed 'dark'). The limiting values of the phase of $q_1^{(s)}(x, y)\exp(2\mathrm{i}r^2 x)$ are 0 and 2ϑ, depending on the direction in which we tend to infinity.

From (4.19) and (4.23), (4.25), one can obtain the following curl representation of the conserved currents in the dark soliton case:

$$J^{(0)} = \mathrm{curl}_2 \ln \frac{A_N^+}{A_N^-} \tag{5.13}$$

$$J^{(1)} = \mathrm{curl}_2 \left(-\frac{\partial_y A_N}{2A_N}\right) \qquad J^{(-1)} = \mathrm{curl}_2 \left(-\frac{\partial_x A_N}{2A_N}\right). \tag{5.14}$$

The N-soliton solution (5.4) depends, when ρ is fixed, on N real parameters ψ_k. The solutions presented below are in some sense 'richer': they are defined up to an arbitrary function. They correspond to the finite nonlinear Schrödinger chains that were discussed in [12], where it is shown that the bounded ($n \geqslant 0$) system (3.2) for $\kappa = -1$ possesses solutions of the form

$$q_n = (-)^n \frac{B_n^+}{B_n} \qquad \bar{q}_n = (-)^n \frac{B_n^-}{B_n} \tag{5.15}$$

where the Toeplitz determinants B_n, $B_n^{\pm}$ are given by

$$B_n \equiv \det|\omega_{j-k}|_{j,k=1,\ldots,N} \qquad B_n^{\pm} \equiv \det|\omega_{j-k\pm 1}|_{j,k=1,\ldots,N} \tag{5.16}$$

and $\omega_{\pm j}$ are related by

$$\omega_{-j} = \bar{\omega}_j. \tag{5.17}$$

One can straightforwardly check that (5.15) is a solution to (2.8) and (2.8) and, hence, to the field equation (2.4), if the ω_j satisfy the *linear* system

$$\mathrm{i}\frac{\partial\omega_j}{\partial x} = \omega_{j+1} - \omega_{j-1} \tag{5.18}$$

$$\frac{\partial\omega_j}{\partial y} = \omega_{j+1} + \omega_{j-1}. \tag{5.19}$$

Note that the ω_j are solutions to the Helmholtz equation

$$\Delta\omega - 4\omega = 0. \tag{5.20}$$

The solution of the system (5.19) satisfying conditions (5.17) can be written as

$$\omega_j = \Omega_j + \bar{\Omega}_{-j} \tag{5.21}$$

where

$$\Omega_j = \int_\Gamma \mathrm{d}\zeta\, \hat{\Omega}(\zeta)\zeta^j \exp\{\mathrm{i}(\zeta^{-1} - \zeta)x + (\zeta^{-1} + \zeta)y\} \tag{5.22}$$

and the function $\hat{\Omega}$ is arbitrary, as well as the contour Γ.

In this way, the functions

$$q_N^{(f)}(x, y) = \frac{B_N^+(x, y)}{B_N(x, y)} \tag{5.23}$$

where the determinants B_N, B_N^+ are given by (5.16) and (5.21), (5.22), for any N, solve the field equation (2.4).

The simplest of these can be written as

$$q_1^{(f)} = \mathrm{i}\frac{\partial}{\partial z} \ln \omega_0(x, y). \tag{5.24}$$

As an example, consider the 'cylindrical' solutions with separable variables. Choosing $\hat{\Omega}(\zeta) = \text{constant}$, one can write them in the cylindrical coordinates

$$x = r\cos\varphi \qquad y = r\sin\varphi \tag{5.25}$$

as follows

$$q_N^{(c)} = \mathrm{e}^{-\mathrm{i}N\varphi} f_N(r) \tag{5.26}$$

with

$$f_N(r) = \frac{\det|I_{j-k+1}(2r)|_{j,k=1,\dots,N}}{\det|I_{j-k}(2r)|_{j,k=1,\dots,N}} \tag{5.27}$$

where I_k are the modified Bessel functions.

A simple analysis of expression (5.27) yields that $f_N(r) \propto r^N$ for small r and that $f_N(r) = 1 - \mathrm{O}(r^{-1})$ for large r.

The curl representation of the conserved currents in the case of 'Toeplitz' solutions (5.23) can be written as

$$\boldsymbol{J}^{(0)} = \mathrm{curl}_2 \ln \frac{B_{N+1}}{B_{N-1}} \tag{5.28}$$

$$\boldsymbol{J}^{(1)} = \mathrm{curl}_2\left(-\frac{\partial_y B_N}{2B_N}\right) \qquad \boldsymbol{J}^{(-1)} = \mathrm{curl}_2\left(-\frac{\partial_x B_N}{2B_N}\right) \tag{5.29}$$

(compare with (5.13), (5.14)).

6. Conclusion

In the present work, we have studied the $O(3,1)$ σ-model using the relation of this model with the well known Ablowitz–Ladik hierarchy. This approach enables us to obtain a set of symmetries and conservation laws, as well as solitons and some other solutions for the field equation. However, the investigation reported above is far from exhaustive. There are a number of questions that have not been answered in the paper.

The existence of an infinite set of the conserved currents (formally, their independence has yet to be proved) and of soliton solutions, as well as the results of the corresponding studies of the $O(4)$ model, clearly indicate that the $O(3,1)$ σ-model is really an integrable model. However, strictly speaking, this conclusion has not been proved in the present paper and, hence, this is still to be achieved.

To confirm the integrability of the model considered it is desirable to develop the corresponding inverse-scattering scheme. This will also provide answers to some problems that are difficult to solve in the framework of the approach used above. So, for example, in section 4, only a few of the conserved currents were presented, which were derived by 'converting' from the symmetries. In principle, we can obtain all of these (since we know an infinite set of the integrals of the Ablowitz–Ladik model, we know an infinite set of the differentiations and, hence, an infinite number of the symmetries, which provides, in principle, an infinite set of the conservation laws) but this procedure is rather cumbersome and provides a closed expression for the generating function, which can be obtained in the framework of the ISM. One other problem that may be solved using the inverse-scattering technique is to obtain the general solution of the field equation (2.4) (see the work by Krichever [14]).

So, the question of developing the inverse-scattering scheme is of primary importance. This problem is rather non-trivial and especially worth investigating. All the more since the established relation between the $O(3,1)$ σ-model and the Ablowitz–Ladik hierarchy ceases to be apparent in the framework of the ISM, since the ISM for the model considered, which is a field model, differs drastically from the ISM for the Ablowitz–Ladik model, which is a discrete model (one can get an insight into the inverse-scattering scheme for the $O(3,1)$ σ-model from the work by Lund [2] devoted to the $O(4)$ case).

Another range of questions that has not been mentioned here is related to the topological aspects of the proposed σ-model: the homotopical classification of solutions, existence of the solutions with finite action, etc.

I now want to mention some models closely related to the one considered. Using the procedure outlined in section 3 and starting from the DNLSE (3.2) with $\kappa = +1$, one can derive the field model described by the Lagrangian

$$\mathcal{L} = \frac{(\nabla q, \nabla \bar{q})}{1 + |q|^2} + 4|q|^2. \tag{6.1}$$

Another remarkable, and rather surprising, feature of the Ablovitz–Ladik hierarchy is its connection with the famous 2D Toda chain. One can straightforwardly check that the quantities p_n defined by $p_n \equiv 1 - |q_n|^2$ (see (3.6)) solve the equation

$$\tfrac{1}{4}\Delta \ln p_n = p_{n-1} - 2p_n + p_{n+1} \tag{6.2}$$

which can be rewritten in terms of the functions u_n, defined by $u_n - u_{n-1} = \ln p_n$, as

$$\tfrac{1}{4}\Delta u_n = \mathrm{e}^{u_{n+1}-u_n} - \mathrm{e}^{u_n - u_{n-1}}. \tag{6.3}$$

To conclude, I want to note the following. The approach used in this paper is, in some sense, an alternative to the traditional ISM and can be useful in some situations. So, for example, the solutions given by (5.23) can hardly be obtained in the framework of the standard version of the ISM for, say, the $O(3, 1)$ model: a model similar to the scheme by Lund [2] (the crucial moment here is the boundary conditions). At the same time, they naturally appear in the framework of the Ablowitz–Ladik hierarchy. As another example, I will mention the 2D Toda chain (6.3). The corresponding inverse-scattering scheme is based on the multidimensional ISM, which is technically rather difficult. So, possessing an alternative to the ISM can be useful when, for example, physical applications of the 2DTL are considered.

Appendix. Principal chiral fields

Consider the Lagrangian of the Euclidean version of the principal chiral-field model

$$\mathcal{L} = \operatorname{tr} g_z (g^{-1})_{\bar{z}} = -\operatorname{tr} g_z g^{-1} g_{\bar{z}} g^{-1} \tag{A.1}$$

where g is a 2×2 matrix. The corresponding Euler–Lagrange equations

$$2g_{z\bar{z}} = g_z g^{-1} g_{\bar{z}} + g_{\bar{z}} g^{-1} g_z \tag{A.2}$$

can be rewritten in terms of the matrices

$$A \equiv g_z g^{-1} \qquad B \equiv g_{\bar{z}} g^{-1} \tag{A.3}$$

as

$$A_{\bar{z}} = -\tfrac{1}{2}[A, B] \qquad B_z = \tfrac{1}{2}[A, B]. \tag{A.4}$$

It follows from (A.4) that

$$\frac{\partial}{\partial \bar{z}} \operatorname{tr} A^n = 0 \qquad \frac{\partial}{\partial z} \operatorname{tr} B^n = 0 \tag{A.5}$$

(note that $\operatorname{tr} A = \operatorname{tr} B = 0$ if $\det g = \text{constant}$).

Hereafter, it will be assumed that

$$\det A = \det B = 1. \tag{A.6}$$

The case $\det A = a^2(z)$, $\det B = b^2(\bar{z})$ can be reduced to (A.6) by means of the substitution

$$\begin{aligned} &A(z, \bar{z}) = a(z) A_1(z_1, \bar{z}_1) \qquad B(z, \bar{z}) = b(\bar{z}) B_1(z_1, \bar{z}_1) \\ &z_1 = \int \mathrm{d}z\, a(z) \qquad \bar{z}_1 = \int \mathrm{d}\bar{z}\, b(\bar{z}). \end{aligned} \tag{A.7}$$

The derivatives A_z, $B_{\bar{z}}$ can be presented as

$$A_z = a_1 C + a_2 [A, C] \tag{A.8}$$

$$B_{\bar{z}} = b_1 C + b_2 [B, C] \tag{A.9}$$

where

$$C \equiv [A, B] \tag{A.10}$$

and the functions $a_{1,2}$, $b_{1,2}$ will be defined below. Multiplying equations (A.8) and (A.9) by C and taking the trace, one can get

$$4a_2 = \frac{\tau_z}{1-\tau^2} \qquad 4b_2 = -\frac{\tau_{\bar{z}}}{1-\tau^2} \tag{A.11}$$

with

$$\tau = \tfrac{1}{2} \operatorname{tr} AB. \tag{A.12}$$

Differentiating (A.8) with respect to $\bar{z}$ and (A.9) with respect to z, and expressing $A_{z\bar{z}}$ and $B_{z\bar{z}}$ using (A.4), one can obtain the following equations:

$$\frac{\partial a_2}{\partial \bar{z}} + 4\tau a_2 b_2 + a_1 b_1 + \tfrac{1}{4} = 0 \tag{A.13}$$

$$\frac{\partial b_2}{\partial z} - 4\tau a_2 b_2 - a_1 b_1 - \tfrac{1}{4} = 0 \tag{A.14}$$

$$\frac{\partial a_1}{\partial \bar{z}} + 4\tau b_2 a_1 - 4a_2 b_1 = 0 \tag{A.15}$$

$$\frac{\partial b_1}{\partial z} - 4\tau a_2 b_1 + 4b_2 a_1 = 0. \tag{A.16}$$

These equations, which are the compatibility conditions for equations (A.4) and (A.8), (A.9), can be rewritten in terms of α defined by

$$\tau = \cos 2\alpha \tag{A.17}$$

as

$$\alpha_{z\bar{z}} - \sin\alpha \cos\alpha\, (1 + 4a_1 b_1) = 0 \tag{A.18}$$

$$\left(a_1 \sin^2\alpha\right)_{\bar{z}} + \left(b_1 \sin^2\alpha\right)_z = 0 \tag{A.19}$$

$$\left(a_1 \cos^2\alpha\right)_{\bar{z}} - \left(b_1 \cos^2\alpha\right)_z = 0. \tag{A.20}$$

After expressing a_1 and b_1 using equation (A.19) as

$$a_1 = \frac{\beta_z}{2\sin^2\alpha} \qquad b_1 = -\frac{\beta_{\bar{z}}}{2\sin^2\alpha} \tag{A.21}$$

the remaining two functions become (1.16) and (1.17).

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